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ON κ-LIKE STRUCTURES WHICH EMBED STATIONARY AND CLOSED UNBOUNDED SUBSETS

James H. SCHMERL *
Department of Mathematics, University of Connecticut, Storrs, CT 06268, USA

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0. Introduction

The only structures which will be considered here are of the form $\mathfrak{A} = (A, <, ...)$, where $<$ is a linear ordering of A. The structure $\mathfrak{A}$ is κ-*like* iff card $(A) = \kappa$ yet every proper initial segment has cardinality $< \kappa$. Suppose that $X \subseteq \kappa$ and that $e : X \to A$ is a function. Then we will say that $e : X \to \mathfrak{A}$ is an *embedding* iff e is order-preserving and continuous. (This means, respectively, that (1) whenever $\mu, \nu \in X$ and $\mu < \nu$, then $e(\mu) < e(\nu)$; and (2) whenever $\nu \in X$ is such that $\nu = \sup(X \cap \nu)$, then for each $a < e(\nu)$ there is $\mu \in X \cap \nu$ such that $a < e(\mu)$.) If $e : X \to \mathfrak{A}$ is an embedding, then $\mathfrak{A}$ *embeds X*.

In this paper we shall be concerned with κ-like structures which embed large subsets of κ. There will be two principal interpretations of the term "large subsets", namely closed unbounded subsets and stationary subsets. A subset $X \subseteq \kappa$ is *closed unbounded* iff $\kappa = \sup(X)$ and whenever $\nu < \kappa$ is such that $\nu = \sup(\nu \cap X)$, then $\nu \in X$. A subset $X \subseteq \kappa$ is *stationary* iff whenever $C \subseteq \kappa$ is closed unbounded, then $C \cap X \neq 0$.

Transfer theorems for κ-like structures which embed large subsets will be given here. Semantic conditions will be developed which guarantee that a theory has these sorts of models. These conditions rely heavily on two hierarchies of subsets: the stationary hierarchy, which generalizes the notion of a stationary subset, and the subtle hierarchy, which is discussed

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by Baumgartner in [2]. It should be remarked that the results contained herein were inspired by a reading of [2], and indeed some of the combinatorial techniques were adapted therefrom.

There are some results concerning κ-like models which embed large subsets of κ that can easily be deduced from known transfer theorems and their proofs. These are discussed in section 1.

Closely related to the problem of finding κ-like structures is the following problem: Given a structure $\mathfrak{A}$, find a proper elementary end-extension $\mathfrak{B}$ of $\mathfrak{A}$. Recall that $\mathfrak{B}$ is an *end-extension* of $\mathfrak{A}$ iff whenever $a \in A$ and $b \in B - A$, then $a < b$. We will say that $\mathfrak{B}$ is a *blunt* end-extension of $\mathfrak{A}$ if it is a proper end-extension and $B - A$ has a least element. Finding blunt elementary end-extension is closely related to the problem of finding κ-like structures which embed large subsets. Notice that if a theory T is such that each model of T has a blunt elementary end-extension, then for each $\kappa > \operatorname{card}(T) + \aleph_0$, there is a κ-like model which embeds κ. This typifies the method used for constructing κ-like models which embed large subsets of κ. In section 2 we will discuss the problem of finding blunt elementary end-extensions.

The stationary and subtle hierarchies are defined in section 3, and some elementary properties and examples are given. The transfer theorems for κ-like structures which embed large subsets are given in section 4. There are theorems for admissible fragments of $L_{\omega_1\omega}$, as well as for ordinary finitary logic. Partition properties will be used in section 5 to show that these transfer theorems are the best possible. A connection with Erdös cardinals is discussed in section 6.

1. Known results [1]

We will extract from known transfer theorems and their proofs some consequences concerning κ-like structures which embed large subsets. Vaught's [15] two-cardinal theorem (as amended by Fuhrken [6]) asserts: If T is a countable theory which has a κ-like model for some regular κ; then it has an ω_1-like model. It is easy to see from Vaught's proof that, using blunt end-extensions, we can get the following result: If T is

[1] It has been pointed out by S. Shelah that most of the observations of section 1 were also made by him in his paper "Generalized quantifiers and compact logic" (Trans. Amer. Math. Soc. 204 (1975) 342–364.)

a countable theory which has a κ-like model which embeds a stationary subset for some regular κ, then it has an ω_1-like model which embeds ω_1. Actually, it suffices to assume the weaker hypothesis that T has a model which has a blunt elementary end-extension. Notice that we can derive from this a simple recursive axiomatization of the class of ω_1-like structures which embed ω_1. For a formula ϕ, let $\phi^{(y)}$ be the relativization of ϕ to the set of predecessors of y. Let Σ be the set of sentences of the form

$$\exists y \, \forall x_0, ..., x_{n-1} \, [\bigwedge_{i<n} x_i < y \to (\phi(\bar{x}) \leftrightarrow \phi^{(y)}(\bar{x}))] \, ,$$

where ϕ is an n-ary formula. Then T has an ω_1-like model which embeds ω_1 iff $T \cup \Sigma$ is consistent.

The previous discussion can be applied to models of set theory. For models of set theory we need to modify slightly our definition of κ-like. We let the relation $<$ in a model $\mathfrak{A} = (A, E)$ of ZF be the restriction of E to the ordinals of $\mathfrak{A}$, and then extend the notion of κ-like structure to admit the situation in which the field of $<$ is not all of A. The obvious modifications concerning other notions (e.g. end-extension, etc.) should also be made. Then it follows that if $\mathfrak{A}$ is any model of ZF, then there is an ω_1-like $\mathfrak{B} \equiv \mathfrak{A}$ which embeds ω_1. This result is easily seen, as in this situation the set Σ is just a formalization of Lévy's Reflection Principle [13].

We can employ considerations with respect to Chang's [3] two-cardinal theorem as we did with Vaught's to get: If κ and λ are regular cardinals such that $2^{<\kappa} = \kappa$ and T has a λ-like model which embeds a stationary subset of λ, then T has a κ^+-like model which embeds a stationary subset of κ^+. The stationary subset of κ^+ we get is $\{\nu < \kappa^+ : \mathrm{cf}(\nu) = \kappa\}$. More generally, if we use Jensen's [8] two-cardinal theorem we get: ($V = L$). If λ is a regular cardinal and T has a λ-like model which embeds a stationary subset of λ, then T has a κ^+-like model which embeds a stationary subset of κ^+. The stationary subset of κ^+ which is involved here is $\{\nu < \kappa^+ : \mathrm{cf}(\nu) = \mathrm{cf}(\kappa)\}$.

The consequence of Vaught's proof of his two-cardinal theorem can be arrived at via Keisler's proof [10]. One can then obtain the stronger result: If T is a countable, complete theory which has a κ-like model which embeds a stationary subset of κ for some regular κ, then there is a consistent, countable $T' \supseteq T$ such that every countable model of T' has a blunt elementary end-extension. For models of set theory, this was announced by Hutchinson [7]. The crucial fact used in this proof is

Fodor's Theorem [6], which we will frequently use in this paper, so we give an explicit statement of it. We first make an important definition. Let α be an ordinal and $X \subseteq \alpha$. A function $f : X \to \alpha$ is *regressive* iff $f(\nu) < \nu$ whenever $\nu \in X$.

Fodor's Theorem. *If κ is a regular cardinal, $X \subseteq \kappa$ is a stationary subset, and $f : X \to \kappa$ is a regressive function, then there is $\nu < \kappa$ such that $\{\mu \in X : f(\mu) = \nu\}$ is a stationary subset of κ.*

2. Blunt end-extensions

It is to be shown in this section that if a theory has a κ-like model which embeds a stationary subset of κ, where κ is strongly inaccessible, then many models of the theory have blunt elementary end-extensions.

Before doing this, let us recall a theorem of Keisler and Morley [12] concerning models of set theory. For a structure $\mathfrak{A}$, we will denote by $\mathrm{cf}(\mathfrak{A})$ the cofinality of the order type of $<^{\mathfrak{A}}$. Their theorem is: If $\mathfrak{A}$ is a model of ZF and $\mathrm{cf}(\mathfrak{A}) = \omega$, then $\mathfrak{A}$ has a proper elementary end-extension.[2]

This theorem has as a corollary a general model-theoretic result. For each similarity type ρ there is a set Σ of sentences (in some larger language) such that: (1) if T has a κ-like model for some strongly inaccessible κ, then $T \cup \Sigma$ is consistent; and (2) every model $\mathfrak{A}$ of Σ for which $\mathrm{cf}(\mathfrak{A}) = \omega$ has a proper elementary end-extension. An analogous result for κ-like structures which embed stationary subsets will be given here.

First we make a digression to remark about a well-known method of extending models. A structure $\mathfrak{A}$ is *tidy* if it becomes Skolemized when expanded by the parametrically definable functions $f : A^n \to A$. An easy way to get tidy expansions of $\mathfrak{A}$ is by adjoining a binary relation W which well-orders A so as to get the structure $(\mathfrak{A}, W)$. Then, not only is $(\mathfrak{A}, W)$ tidy, but also any expansion of it is. With this in mind, we define $\alpha(\rho, W)$, where ρ is a similarity type and W is a binary relation symbol of ρ, to be

[2] Keisler and Morley [12] actually have a stronger conclusion – that $\mathfrak{A}$ has arbitrarily large elementary end-extensions. We remark that for any model of ZF, if it has an elementary end-extension, then it has arbitrarily large ones. For, if $\mathfrak{A}_0$ is a proper elementary end-extension of $\mathfrak{A}$, then by the Reflection Principle, there is $\mathfrak{B}$ such that $\mathfrak{A} \neq \mathfrak{B} \prec \mathfrak{A}_0$ and $\mathrm{cf}(\mathfrak{B}) = \omega$. Then $\mathfrak{B}$ has arbitrarily large elementary end-extensions. This gives an affirmative answer to a question in Chang–Keisler [4, p. 450].

the set of universal closures of the following ρ-formulas:

(1) $\exists u\, \phi(\bar{x}, u) \to \exists u[\phi(\bar{x}, u) \wedge \forall v(\phi(\bar{x}, v) \to W(u, v))]$, for each $m < \omega$ and each $(m+1)$-ary ρ-formula ϕ.

(2) $(W(x, y) \wedge W(y, x)) \to x = y$.

Then if $\mathfrak{A}$ is a ρ-structure which is a model of $\alpha(\rho, W)$, then $\mathfrak{A}$ is tidy; and if W well-orders A, then $\mathfrak{A}$ is a model of $\alpha(\rho, W)$.

Now let $\mathfrak{A}$ be a tidy structure. Let U be an ultrafilter in the Boolean algebra of parametrically definable subsets of $\mathfrak{A}$, and let $V \supseteq U$ be an ultrafilter in $P(A)$, the power-set of A. Then let $\mathfrak{A}^U$ be the restriction of the ultrapower $\mathfrak{A}^A_V$ to the parametrically definable functions of $\mathfrak{A}$. Notice that $\mathfrak{A}^U$ is independent of V. There is a canonical elementary embedding of into $\mathfrak{A}^U$; so making the natural identifications, we have $\mathfrak{A} \prec \mathfrak{A}^U$.

We now return to κ-like structures which embed a stationary subset $X \subseteq \kappa$, where κ is strongly inaccessible. In such a case, the set of cardinals in X is stationary; in fact, the set of strong limit cardinals in X is stationary. By Fodor's Theorem there are two possibilities: either (1) the set of regular cardinals in X is stationary, or (2) the set of singular cardinals is stationary. The results of this section divide naturally according to these two possibilities.

We first investigate the case (1) in which the stationary subset of κ consists of regular cardinals (thus implying that the cardinal κ is mahlo).

Let ρ be any similarity type. We wish to define a set Σ_1 of sentences (in some larger language). Let W be a new binary relation symbol, U_n a new binary relation symbol for each $n < \omega$, and R_n a new n-ary relation symbol for each $n < \omega$. For each $n < \omega$, let $\rho_n = \{W, R_0, R_1, ...\} \cup \{U_0, ..., U_n\}$, and let $\rho_\omega = \bigcup\{\rho_n : n < \omega\}$ and $\rho' = \rho \cup \rho_\omega$. We will now define the set Σ_1 to be the set of universal closures of the following ρ'-formulas.

(1) $\alpha(\rho', W)$.

(2) $\exists y\, \forall x_0, ..., x_{n-1}(\phi(\bar{x}) \leftrightarrow R_n(\bar{x}, y))$, for each $n < \omega$ and each n-ary ρ-formula ϕ.

(3) $\forall u, w \exists v(w < v \wedge U_n(u, v))$, for each $n < \omega$.

(4)
$$\begin{aligned}\exists t\, [\forall v(U_{m+1}(t, v) \to U_m(u, v)) \wedge \\ \wedge \forall x_0, ..., x_{n-1}((\bigwedge_{i<n} x_i < x \wedge \forall v \exists w\, \phi(\bar{x}, v, w)) \to \\ \to (\exists w\, \forall v(U_{m+1}(t, v) \to \phi(\bar{x}, v, w)) \wedge \\ \vee\ \forall v \exists w(U_{m+1}(t, v) \to v \leq w \wedge \phi(\bar{x}, v, w))))] ,\end{aligned}$$

for each $m, n < \omega$ and each $(n+2)$-ary ρ_m-formula.

We shall prove two lemmas about the set Σ_1.

Lemma 2.1. *Let T be a ρ-theory such that* $\mathrm{card}(T) \leqslant \kappa$, *where κ is strongly inaccessible. Suppose that $\mathfrak{A}$ is a κ-like model of T which embeds a stationary subset of κ consisting of regular cardinals. Then $\mathfrak{A}$ can be expanded to a model of Σ_1.*

Proof. Suppose that T, κ, and $\mathfrak{A}$ satisfy the hypotheses of the lemma. Let $X \subseteq \kappa$ be a stationary subset consisting of regular cardinals, and let $e : X \to \mathfrak{A}$ be an embedding. We can make the further assumption about X that it consists only of strongly inaccessible cardinals, and that for each $\lambda \in X$ and each $a < e(\lambda)$ the set of predecessors of a has cardinality $< \lambda$.

Let W be any well-ordering of A. Then any expansion of $(\mathfrak{A}, W)$ to a ρ'-structure will satisfy (1). It is easy to find $R_0, R_1, \ldots$ such that $(\mathfrak{A}, W, R_0, R_1, \ldots)$ is a model of the sentences (2). (It is here where we use the fact that $\mathrm{card}(T) \leqslant \kappa$.) Now we must find $U_0, U_1, \ldots$. Let $U_0 = A \times A$. We proceed by induction. Suppose that $U_0, \ldots, U_m$ have been defined so that $\mathfrak{A}_m = (A, W, R_0, R_1, \ldots, U_0, \ldots, U_m)$ is a model of all ρ_m-sentences in Σ_1, and that for each $a \in A$, the set $X_a = \{\lambda \in X : \mathfrak{A}_m \models U_m(a, e(\lambda))\}$ is a stationary subset of κ.

To get U_{m+1}, let $g : A^2 \to A$ be a bijection. For each $x, a \in A$, set $b = g(x, a)$, and let $\mathcal{F}_b$ be the set of all functions $f : A \to A$ such that f is first-order definable in $\mathfrak{A}_m$ allowing only parameters $< x$. Now it is not hard to conclude from Fodor's Theorem, using that $\lambda \in X_a$ is strongly inaccessible, that there is a stationary set $Y_b \subseteq X_a$ such that whenever $f \in \mathcal{F}_b$ and f is not constant on $e''Y_b$ then $e(\lambda) \leqslant f(e(\lambda))$ for each $\lambda \in Y_b$. Then define

$$U_{m+1} = \{(b, e(\lambda)) : b \in A \quad \text{and} \quad \lambda \in Y_b\},$$

and this definition clearly suffices. Therefore, we get that $(\mathfrak{A}, W, R_0, \ldots, U_0, \ldots)$ is a model of Σ_1. □

The following corollary handles the situation when $\mathrm{card}(T)$ is large.

Corollary 2.2. *Let T be a ρ-theory which has a κ-like model which embeds a stationary subset consisting of regular cardinals, where κ is strongly inaccessible. Then $T \cup \Sigma_1$ is consistent.* □

Lemma 2.3. *If $\mathfrak{A}$ is a model of Σ_1 and $\mathrm{cf}(\mathfrak{A}) = \omega$, then $\mathfrak{A}$ has a blunt elementary end-extension.*

Proof. Let $b_0, b_1, \ldots$ be a cofinal sequence in $\mathfrak{A}$. Let $\langle \phi_n(\bar{x}, v, w) : n < \omega \rangle$ be a list of all ρ_ω-formulas in the variables $v, w, x_0, x_1, \ldots$ such that each ϕ_n is a ρ_n-formula. (To be safe, let's assume that each formula occurs infinitely often in this list.) We will get a sequence $\langle a_n : n < \omega \rangle$ of elements of A by induction. Choose $a_0 \in A$ arbitrarily. Now suppose we have a_n. Let a_{n+1} be the element guaranteed by sentences (4). That is, in (4) let $x = b_n$, $u = a_n$ and $\phi = \phi_n$. Then set $a_{n+1} = t$.

We can conclude from (4) that

$$U = \{\{b \in A : \mathfrak{A} \models U_n(a_n, b)\} : n < \omega\}$$

actually is an ultrafilter in the Boolean algebra of parametrically definable subsets of $\mathfrak{A}$, and from (3) we see that it is non-principal. Further, if $B \in U$ and $f : A \to A$ is a parametrically definable function, then there is $B_0 \in U$ such that $B_0 \subseteq B$ and either f is constant on B_0 or $b < f(b)$ for each $b \in B_0$ It is now easy to check that $\mathfrak{A}^U$ is a blunt elementary end-extension of $\mathfrak{A}$, the least new element of $\mathfrak{A}^U$ being the identity function of $\mathfrak{A}$. □

We will now consider the case (2) in which the stationary subset of κ consists only of singular cardinals.

Let ρ be any similarity type. We wish to define a set Σ_2 of sentences. Let W be a new binary predicate symbol, let S and I be new unary predicate symbols, let F be a new binary function symbol, and U_n a new binary relation symbol for each $n < \omega$. Also, for each $n < \omega$ let R_n be a new n-ary relation symbol. For each $n < \omega$ let $\rho_n = \{W, S, I, F, R_0, R_1, \ldots\} \cup \{U_0, \ldots, U_n\}$, and let $\rho_\omega = \bigcup\{\rho_n : n < \omega\}$ and $\rho' = \rho \cup \rho_\omega$. We will now define the set Σ_2 to be the set of universal closures of the following ρ'-formulas together with sentences (1)–(3) in the definition of Σ_1:

(5) $\forall u, v(U_n(u, v) \to S(v))$, for each $n < \omega$.

(6) $S(x) \to [(\forall y < x)\exists z(I(z) \wedge y < F(z, x) < x) \wedge$

$$\wedge \forall z(I(z) \to F(z,x) < x) \wedge \forall z_0, z_1((I(z_0) \wedge I(z_1)$$

$$\wedge z_0 < z_1) \to F(z_0, x) < F(z_1, x))] .$$

$$(7)\quad \exists t\,[\forall v(U_{m+1}(t,v)\to U_m(u,v))\wedge \wedge\forall x_0,\dots,x_{n-1}((\bigwedge_{i<n} x_i<x\wedge\forall v\,\exists w\,\phi(\bar{x},v,w))\to \to(\exists w\,\forall v\,(U_{m+1}(t,v)\to\phi(\bar{x},v,w))\wedge \vee\,\forall v\,\exists w\,(U_{m+1}(t,v)\to F(z,v)\leqslant w\wedge\phi(\bar{x},v,w))))]$$

for each $m, n<\omega$ and each $(n+2)$-ary ρ_m-formula.

The reader should compare the sets Σ_1 and Σ_2, noting especially the distinction between (4) and (7). The statements and proofs of the following lemmas are analogues to the corresponding ones for Σ_1.

Lemma 2.4. *Let T be a ρ-theory such that* $\mathrm{card}(T)\leqslant\kappa$, *where κ is strongly inaccessible. Suppose that $\mathfrak{A}$ is a κ-like model of T which embeds a stationary subset of κ consisting of singular cardinals. Then $\mathfrak{A}$ can be expanded to a model of Σ_2.*

Proof. Suppose that T, κ, and $\mathfrak{A}$ satisfy the hypotheses of the lemma. Let $X\subseteq\kappa$ be a stationary subset consisting of singular cardinals, and let $e: X\to\mathfrak{A}$ be an embedding. Without loss of generality we can make some further assumptions about X. First, we can assume that it consists just of strong limit cardinals and then by Fodor's Theorem we can assume that each $\lambda\in X$ has the same cofinality, say γ. Finally, as in case (1), we can assume that whenever $\lambda\in X$ and $a<e(\lambda)$, then the set of predecessors of a has cardinality $<\lambda$.

Let W be any well-ordering of A, let $S=e''X_0$ where X_0 is the set of limit points of X, and let $I\subseteq A$ be any set of order type γ. Choose the binary function $F: A^2\to A$ such that for any $a\in S$, the function $g: I\to A$ defined by $g(z)=F(z,a)$ is a strictly increasing, cofinal function into the set of predecessors of a. It is easy to find $R_0, R_1, \dots$ such that $(\mathfrak{A}, W, S, I, F, R_0, R_1, \dots)$ is a model of the sentences (2). Set $U_0=A\times S$. We proceed by induction. Suppose that $U_0,\dots,U_m$ have been defined so that $\mathfrak{A}=(A, W, S, I, F, R_0, R_1, \dots, U_0,\dots,U_m)$ is a model of all ρ_m-sentences in Σ_2, and that for each $a\in A$, the set $X_a=\{\lambda\in X:\mathfrak{A}_m\models U_m(a,e(\lambda))\}$ is a stationary subset of κ.

To get U_{m+1}, let $g: A^2\times I\to A$ be a bijection. Choose $x, a\in A$ and $z\in I$, and then set $b=g(x,a,z)$. Let $\mathcal{F}_b$ be the set of all functions $f: A\to$

A such that f is first-order definable in $\mathfrak{A}_m$ allowing only parameters $< x$. Now we can use Fodor's Theorem and the fact that each $\lambda \in X$ is a strong limit cardinal to conclude that there is a stationary set $Y_b \subseteq X_a$ such that whenever $f \in \mathcal{F}_b$ and f is not constant on $e''Y_b$, then $F(z, e(\lambda)) \leqslant f(e(\lambda))$ for each $\lambda \in Y_b$. Then define

$$U_{m+1} = \{(b, e(\lambda)) : b \in A \quad \text{and} \quad \lambda \in Y_b\} .$$

It is easy to see that $(\mathfrak{A}, W, S, I, F, R_0, R_1, \ldots, U_0, U_1, \ldots)$ is a model of Σ_2. □

Corollary 2.5. *Let T be a ρ-theory which has a κ-like model which embeds a stationary subset of κ consisting of singular cardinals, where κ is strongly inaccessible. Then $T \cup \Sigma_2$ is consistent.* □

Lemma 2.6. *If $\mathfrak{A}$ is a model of Σ_2 and* $\mathrm{cf}(\mathfrak{A}) = \mathrm{cf}(I^{\mathfrak{A}}) = \omega$, *then $\mathfrak{A}$ has a blunt elementary end-extension.*

Proof. Let $b_0, b_1, \ldots$ be a cofinal sequence in $\mathfrak{A}$, and let $c_0, c_1, \ldots$ be a cofinal sequence in I. Let $\langle \phi_n(\bar{x}, v, w) : n < \omega \rangle$ be a list of all ρ_ω-formulas in the variables $v, w, x_0, x_1, \ldots$ such that each ϕ_n is a ρ_n-formula. We will get a sequence $\langle a_n : n < \omega \rangle$ of elements of A by induction. Choose $a_0 \in A$ arbitrarily. Let a_{n+1} be the element guaranteed by sentences (7). That is, in (7) let $x = b_n$, $u = a_n$, $z = c_n$ and $\phi = \phi_n$. Then set $a_{n+1} = t$. Let U be as in the proof of Lemma 2.3. Then ^{U} is the desired model. □

3. The stationary and subtle hierarchies

We are going to define two hierarchies of sets of ordinals which extend the notion of a stationary subset.

Let κ be a cardinal, α an ordinal and $X \subseteq \kappa$. The first hierarchy we define is the stationary hierarchy. By induction on α we define when X is an α-stationary subset of κ. X is *0-stationary* iff X is a stationary subset of κ and κ is inaccessible. If $\alpha > 0$, then X is *α-stationary* iff for each $\beta < \alpha$, the set

$$\{\lambda < \kappa : X \cap \lambda \text{ is a } \beta\text{-stationary subset of } \lambda\}$$

is a stationary subset of κ. We say that X is a *strongly α-stationary* subset of κ iff X is α-stationary and κ is strongly inaccessible.

The other hierarchy we will need is the subtle hierarchy. Again by induction on α, we define when X is an α-subtle subset of κ. X is 0-*subtle* iff X is a stationary subset of κ and κ is inaccessible. If $\alpha > 0$, then X is α-subtle iff whenever $\beta < \alpha$ and $\langle S_\nu : \nu \in X \rangle$ is such that $S_\nu \subseteq \nu$, then

$$\{\lambda \in X \{\nu \in X \cap \lambda : S_\nu = \nu \cap S_\lambda\} \text{ is } \beta\text{-subtle}\}$$

is a stationary subset of κ.

The use of the term "hierarchy" is justified by the following facts. If $\alpha < \beta$ and X is a β-stationary [β-subtle] subset of κ, then X is also an α-stationary [α-subtle] subset of κ. If β is a limit ordinal, then X is β-stationary [β-subtle] iff X is α-stationary [α-subtle] for each $\alpha < \beta$.

If we consider κ as a subset of κ, then κ is α-stationary iff κ is α-inaccessible (as defined in [17]). Indeed, if κ has an α-stationary subset at all, then κ is α-inaccessible. If κ is α-inaccessible, then an example of an α-stationary subset, which we will use later on, is the set $\{\nu < \kappa : \mathrm{cf}(\nu) = \omega\}$. Other examples of α-stationary sets can be given by considering weakly compact cardinals. ([21] is the reference to consult for the various equivalent definitions of weakly compact.)

Proposition 3.1. *If κ is weakly compact, then every stationary subset of κ is a κ-stationary subset.*

Proof. We use the Π^1_1-indescribability of κ. It is not hard to see that there is a Π^1_1 sentence σ such that whenever $Y \subseteq \lambda$ and $\alpha < \lambda$, then $(R(\lambda), Y, \alpha) \models \sigma$ iff Y is an α-stationary subset of λ. Suppose now that $X \subseteq \kappa$ is α-stationary, where $\alpha < \kappa$. Then $(R(\kappa), X, \alpha) \models \sigma$. For any closed unbounded $C \subseteq \kappa$ there is $\lambda \in C$ such that $(R(\lambda), X \cap \lambda, \alpha) \models \sigma$. Thus X is $(\alpha + 1)$-stationary. Therefore, the least α such that X is α-stationary is $\alpha = \kappa$. □

Jensen [8] has shown, under the assumption of $V = L$, that each regular cardinal which is not weakly compact has a stationary subset which is not 1-stationary, and this set can be chosen to consist of cardinals of cofinality ω.

The notion of a subtle cardinal was introduced in [9]. Indeed, it is not hard to see that κ is a 1-subtle subset of κ if κ is subtle. Thus, although it is possible for a cardinal κ to be κ-stationary yet not strongly inaccessible, every 1-subtle cardinal is strongly inaccessible. Baumgartner [2] defined n-subtle sets for $n < \omega$. Although his definition appears different from our definition of α-subtle (for finite α), it will be seen in section 5 to be equivalent. We can give examples of subsets $X \subseteq \kappa$ which are κ-subtle.

Proposition 3.2. *If κ is a measurable cardinal and U a normal ultrafilter of κ, then each $X \in U$ is a κ-subtle subset of κ.*

Proof. The following is a well-known property of normal ultrafilters: If σ is a Π^1_1 sentence and $A \subseteq \kappa$ is such that $(R(\kappa), A) \models \sigma$, then for any $X \in U$ there is $\lambda \in X$ such that $(R(\lambda), A \cap \lambda) \models \sigma$. In particular, each $X \in U$ is 0-subtle. There is a Π^1_1 sentence σ such that whenever $Y \subseteq \lambda$ and $\alpha < \lambda$, then $(R(\lambda), Y, \alpha) \models \sigma$ iff Y is α-subtle. It will suffice to show that if each $X \in U$ is α-subtle, then each $X \in U$ is $(\alpha+1)$-subtle. So, suppose each set in U is α-subtle, and let $X \in U$. Let $\langle S_\nu : \nu \in X \rangle$ be such that each $S_\nu \subseteq \nu$, and let $C \subseteq \kappa$ be closed unbounded. It is known [9] that there is $Z \subseteq X \cap C$ such that $Z \in U$ and whenever $\nu, \mu \in Z$ with $\nu < \mu$, then $S_\nu = \nu \cap S_\mu$. Since Z is α-subtle, there is $\lambda \in Z$ such that $Z \cap \lambda$ is an α-subtle subset of λ. Hence X is $(\alpha+1)$-subtle. □

By indescribability it is clear that there are smaller cardinals which have κ-subtle subsets. Indeed, by absoluteness, the least cardinal κ which is κ-subtle, is κ-subtle in L. Further examples are given in section 6.

The previous examples show that we are not dealing with a vacuous situation.

We note the following simple property of a subset $X \subseteq \kappa$: the set X is α-stationary [α-subtle] iff whenever $C \subseteq \kappa$ is closed unbounded, then $C \cap X$ is α-stationary [α-subtle].

We conclude this section with the following proposition which will be needed later on.

Proposition 3.3. *If $\alpha > 0$ and $X \subseteq \kappa$ is α-subtle, then $X_0 = \{\nu \in X : \nu$ is strongly inaccessible$\}$ is α-subtle.*

Proof. The hardest case occurs when $\alpha = 1$ (whereby α-subtle is the same as subtle), and this is proved by Baumgartner [2, Prop. 2.5]. It follows from this that for any λ the set $\{\nu < \lambda : \nu$ is not strongly inaccessible$\}$ is not 1-subtle. Now suppose that $\beta > 1$ and that the Proposition is true whenever $0 < \alpha < \beta$. To see that it's true when $\alpha = \beta$, let $\langle S_\nu : \nu \in X_0 \rangle$ be such that each $S_\nu \subseteq \nu$. Let $\langle T_\nu : \nu \in X - X_0 \rangle$ demonstrate that $X - X_0$ is not 1-subtle, also making sure that $0 \in T_\nu$. For $\nu \in X_0$ define

$$T_\nu = \{1 + \mu : \mu \in S_\nu\} .$$

Then whenever $\lambda \in X$, $0 < \beta < \alpha$ and $Y = \{\nu \in X \cap \lambda : T_\nu = \nu \cap T_\lambda\}$ is β-subtle, then it is easily seen that $\lambda \in X_0$ and $Y = \{\nu \in X_0 \cap \lambda : S_\nu = \nu \cap S_\lambda\}$. Hence, X being β-subtle implies that X_0 also is. □

4. The transfer theorems

We will find in this section semantic conditions on a theory T so that for each $\kappa > \mathrm{card}(T) + \aleph_0$ there is a κ-like model of T which embeds a stationary subset of κ. We will do the same thing for models which embed closed unbounded sets.

To construct the models which embed stationary subsets, we will use the results of section 3, producing κ-like structures which embed $\{\nu < \kappa : \mathrm{cf}(\nu) = \omega\}$.

Theorem 4.1. *Let T be a theory such that for each $n < \omega$ there is a κ-like model of T which embeds a strongly n-stationary subset of κ. Then for each $\lambda > \mathrm{card}(T) + \aleph_0$, there is a model of T which embeds $\{\nu < \lambda : \mathrm{cf}(\nu) = \omega\}$.*

Proof. We will only outline the proof, which relies heavily on the results of [20]. In [20] the following is proved: If T is a theory such that for each $n < \omega$ there is a strong n-inaccessible κ and a κ-like model of T, then T has a λ-like model whenever $\lambda > \mathrm{card}(T) + \aleph_0$. This result is proved in two stages. A set Σ of sentences such that $\mathrm{card}(\Sigma) = \mathrm{card}(T) + \aleph_0$ is defined which has the following two properties. (1) If for each $n < \omega$ there is a strong n-inaccessible κ and a κ-like model of T, then $T \cup \Sigma$ is consistent. (2) Any model of Σ has a proper elementary end-extension. Now this set Σ and either the set Σ_1 or Σ_2 (whichever is appropriate) can be dove-tailed into one set Σ_3, where $\mathrm{card}(\Sigma_3) = \mathrm{card}(T) + \aleph_0$, satisfying:

(1) If T is a theory such that for each $n < \omega$ there is a κ-like model of T which embeds a strongly n-stationary subset of κ, then $T \cup \Sigma_3$ is consistent.

(2) Any model of Σ_3 has a proper elementary end-extension.

(3) If $\mathfrak{A}$ is a model of Σ_3 such that $\mathrm{cf}(\mathfrak{A}) = \omega$ (and also $\mathrm{cf}(I^{\mathfrak{A}}) = \omega$ if applicable), then $\mathfrak{A}$ has a blunt elementary end-extension.

Now it is easy to see how to conclude the proof of the theorem. □

Corollary 4.2. *If T has a κ-like model which embeds a stationary subset of κ, where κ is weakly compact, then T has a λ-like model which embeds a stationary subset whenever* $\lambda > \mathrm{card}(T) + \aleph_0$.

Proof. Combine Proposition 3.1 and Theorem 4.1. □

Theorem 4.1 can be extended to infinitary languages. For any admissible set A let $L_A = A \cap L_{\infty\omega}$. (The book [1] is a fine source of information.) Associated with each admissible set is an ordinal $h(A)$ which is intimately related to the Hanf number of L_A. Let $\preceq$ be a binary relation symbol, and let us say that a structure $\mathfrak{A}$ is *well-ordered* or *non-well-ordered* according as $\preceq$ is well-ordered or not. If $\mathfrak{A}$ is well-ordered, and in fact $\preceq$ has order type α, then we will say that $\mathfrak{A}$ is *α-ordered.* Now we define $h(A)$ to be the least ordinal α with the following property: Whenever σ is a sentence in L_A such that for each $\beta < \alpha$ there is a β-ordered model of σ, then σ has a non-well-ordered model. In case that Barwise Compactness holds for the language L_A (for example, if A is countable), then $h(A) = A \cap \mathrm{Ord}$. (See [1] for a thorough discussion of $h(A)$.)

In [20] the relationship between $h(A)$ and κ-like models is shown. (Actually, in [20] results are stated in terms of omitting types, but the translation to L_A presents no real difficulty: just use the procedure of [14].) It is shown there that if σ is a sentence of L_A and for each $\alpha < h(A)$ there is a κ-like model of σ where κ is strongly n-inaccessible, then σ has a λ-like model whenever $\lambda > \mathrm{card}(A)$. The statement is proved by the method of indiscernibles, although an alternate proof using the technique described above for finitary σ is available. See [20, Remark 2.8], or, for a more elaborate discussion, consult [16, Chapter 6]. We thus get the following extension of Theorem 4.1 to infinitary languages.

Theorem 4.3. *Let A be an admissible set and σ a sentence of L_A such that for each $\alpha < h(A)$ there is a κ-like model of σ which embeds a strongly α-stationary subset X of κ. Furthermore, suppose that either* (i) *X consists of regular cardinals*; (ii) *each $\nu \in X$ has cofinality ω; or* (iii) *A is countable. Then for each* $\lambda > \mathrm{card}(A)$ *there is a λ-like model of σ which embeds* $\{\nu < \lambda : \mathrm{cf}(\nu) = \omega\}$.

Remark. In the theorem, if X consists of singular cardinals, we must guarantee that we can find a model $\mathfrak{A}$ of σ for which $\mathrm{cf}(I^{\mathfrak{A}}) = \omega$. This is the reason for the inclusion of conditions (ii) and (iii).

We now turn our attention to finding structures which embed closed unbounded sets.

Theorem 4.4. *Let T be a theory such that for each $n < \omega$ there is a κ-like model of T which embeds an n-subtle subset of κ. Then for each $\lambda >$* $\mathrm{card}(T) + \aleph_0$, *there is a λ-like model of T which embeds λ.*

The proof represents only a small change from the proof of Theorem 1 of [20]. We will assume that the reader has [20] available for reference.

For a similarity type ρ let us define a set Σ_4 of sentences (in a larger language which includes the new binary relation symbols $W, R_0, R_1, \ldots$). This set Σ_4 will be the same as the set Γ of [20], except that (6) should be replaced by the following:

$$(6') \quad \forall v \exists y\, \phi(\bar{x}, v, y) \to \exists u [\exists y\, \forall v (R_i(u,v) \to \phi(\bar{x}, v, y)) \vee (\forall v (\exists y \geq v) \\ (R_i(u,v) \to \phi(\bar{x}, v, y))],$$

for each $n, i < \omega$ and each $(n+2)$-ary ρ_i-formula ϕ.

Theorem 4.4 is proved by proving the following two lemmas, which correspond to Lemmas 1.1 and 1.2 of [20].

Lemma 4.5. *Every model of Σ_4 has a blunt elementary end-extension.*

Proof. The construction of the blunt extension proceeds exactly as the construction in Lemma 1.1 of [20]. Notice that (6′) guarantees the extension will have a least new element, namely the identity function. □

Lemma 4.6. *If T is a ρ-theory such that for each $n < \omega$ there is a κ-like model of T which embeds an n-subtle subset of κ, then $T \cup \Sigma_4$ is consistent.*

Proof. Generally, the proof follows the lines of Lemma 1.2 of [20]. We isolate in the following statement the only place where the difference occurs.

Suppose $\mathfrak{A}$ is a κ-like structure, where $\kappa > \mathrm{card}(\rho)$ and suppose $e : X \to \mathfrak{A}$ embeds the $(n+1)$-subtle subset X of κ. Then there is $\lambda \in X$ such that $\mathfrak{A} \restriction \{a \in A : a < e(\lambda)\} \prec \mathfrak{A}$, and if $D \subseteq A$ is definable allowing only parameters $< e(\lambda)$ such that $e(\lambda) \in D$, then $\{y \in X \cap \lambda : e(y) \in D\}$ is n-subtle.

To prove the above statement, notice first that κ is strongly inaccessible, and that by Proposition 3.3 we can assume without loss of generality that X is a set of strongly inaccessible cardinals. Therefore, there is a closed unbounded $C \subseteq \kappa$ such that if $\lambda \in C$ then $\mathfrak{A}_\lambda = \mathfrak{A} \upharpoonright \{a \in A : a < e(\nu)$ for some $\nu < \lambda\}$ is relatively saturated in $\mathfrak{A}$. Then $C \cap X$ is still $(n+1)$-subtle. For each $\lambda \in C \cap X$ define the function $f^\lambda : \lambda \to C \cap X \cap \lambda$ by induction in the following way. Let $f^\lambda(\nu)$ be the least $\mu \in C \cap X \cap \lambda$ such that $e(\lambda)$ and $e(\mu)$ realize the same type in the structure $(\mathfrak{A}, a)_{a \in I}$, where $I = \{a \in A : a \leqslant e(f^\lambda(\xi))$ for some $\xi < \nu\}$. Due to relative saturation, f^λ is well-defined. Let $S_\lambda = \{f^\lambda(\nu) : \nu < \lambda\}$. Notice that if $\nu \in S_\lambda$, then $S_\nu = \nu \cap S_\lambda$. Thus there is $\lambda \in X \cap C$ such that S_λ is n-subtle. It is clear that λ works. $\square$

The previous theorem can be generalized to admissible fragments.

Theorem 4.7. *Let A be an admissible set and σ a sentence of L_A such that for each $\alpha < h(A)$ there is a κ-like model of σ which embeds an α-subtle subset of κ. Then for each $\lambda >$ card(A) there is a λ-like model of σ which embeds λ.*

5. Optimality of transfer theorems

In this section we will show that the transfer theorems of the previous section are the best possible. In order to demonstrate this optimality we will resort to some partition properties closely related to the stationary and subtle hierarchies. We will need to start with a series of definitions (cf. [17], [18]).

For a set X we denote by $[X]^n$ the set of subsets of X with exactly n elements, and we denote by $S_\omega(X)$ the set of finite subsets of X. It will be convenient to have a pairing function available. For a cardinal κ we will make use of the pairing function $p : S_\omega(\kappa) \times \kappa \to \kappa$, where $p''(S_\omega(\lambda) \times \lambda) \subseteq \lambda$ for each infinite cardinal $\lambda < \kappa$.

Baumgartner [2] gave an extension of the notion of a regressive function which we will use. Let ν be an ordinal and $X \subseteq \nu$. A function $f : S_\omega(X) \to \nu$ is *regressive* iff whenever $\mu \in A \in S_\omega(X)$, then $f(A) < \mu$. Suppose that $f : S_\omega(X) \to \nu$ is any function. We say that $Y \subseteq X$ is *f-homogeneous* iff whenever $n < \omega$ and $A, B \in [Y]^n$, then $f(A) = f(B)$. For a weaker notion, we say that $Y \subseteq X$ is *f-semihomogeneous* iff whenever

$A, B \in S_\omega(Y)$ are such that $A - \{\min(A)\} = B - \{\min(B)\}$, then $f(A) = f(B)$.

We define a partial order $\lhd$ on $S_\omega(\nu)$ so that if $A, B \in S_\omega(\nu)$, then $A \lhd B$ iff there is $\alpha \in A$ such that $B = \{\beta \in A : \alpha < \beta\}$. Thus $(S_\omega(\nu), \lhd)$ is well-founded; in fact, each chain of elements is finite. The rank of an element $A \in S_\omega(\nu)$ is $\min(A)$. (Here, and subsequently, whenever $A \in S_\omega(\nu)$ and $A = 0$, then we will understand ν for $\min(A)$.) The notion of homogeneous and semihomogeneous will now be extended. Let $X \subseteq \nu$ and $f : S_\omega(X) \to \nu$. Then the function $F : S_\omega(\alpha) \to X$ is *f-homogeneous* [*f-semihomogeneous*] iff the following hold:

(1) if $A, B \in S_\omega(\alpha)$ and $A \lhd B$, then $F(A) < F(B)$;

(2) if $L \subseteq S_\omega(\alpha)$ is linearly ordered by $\lhd$, then $F''L$ is f-homogeneous [f-semihomogeneous].

The partition properties we shall be interested in are the ones characterized in the following two theorems.

Theorem 5.1. *There is a first-order sentence σ_1 with the following property: Whenever κ is inaccessible and $X \subseteq \kappa$, then σ_1 has a κ-like α-ordered model which embeds X iff there is a closed unbounded $C \subseteq \kappa$ and a regressive $f : S_\omega(X \cap C) \to \kappa$ for which there exists no f-semihomogeneous $F : S_\omega(1 + \alpha) \to X \cap C$.*

Theorem 5.2. *There is a first-order sentence σ_2 with the following property: Whenever κ is inaccessible and $X \subseteq \kappa$, then σ_2 has a κ-like α-ordered models which embeds X iff there is a closed unbounded $C \subseteq \kappa$ and a regressive $f : S_\omega(X \cap C) \to \kappa$ for which there exists no f-homogeneous $F : S_\omega(1 + \alpha) \to X \cap C$.*

The proofs of Theorems 5.1 and 5.2 parallel the proofs of analogous theorems. See Lemma 3.1 of [17], or, for a more succinct version, Theorem 4.1 of [18].

We will refer to the sentences described in Theorems 5.1 and 5.2 as σ_1 and σ_2 respectively. Fodor's Theorem implies that whenever κ is inaccessible and $X \subseteq \kappa$, then σ_1 has a κ-like 0-ordered model which embeds X iff X is not a stationary subset of κ. The same is true for σ_2. We give some further examples.

Proposition 5.3. *If κ is weakly compact, $X \subseteq \kappa$ is stationary and $\alpha < \kappa$, then there is no κ-like α-ordered model of σ_1 which embeds X.*

Proof. By induction on $\alpha < \kappa$ we show that whenever $X \subseteq \kappa$ is stationary and $f : S_\omega(X) \to \kappa$ is regressive, then there is an f-semihomogeneous $F : S_\omega(1+\alpha) \to X$. Suppose it is true for each $\alpha < \beta$, where $\beta < \kappa$. By Fodor's Theorem there is a stationary $Y \subseteq X$ such that f is constant on $[Y]^1$, and we can assume that Y consists only of infinite cardinals. Using the Π^1_1-indescribability of κ, there is λ such that for each $\alpha < \beta$ and each regressive $g : S_\omega(Y \cap \lambda) \to \lambda$ there is a g-semihomogeneous $G : S_\omega(1+\alpha) \to Y$. Select $\lambda_0 \in Y$ such that $\lambda_0 > \lambda$. Define $g : S_\omega(Y) \to \lambda$ by $g(A) = p(f(A), f(A \cup \{\lambda_0\}))$. For each $\alpha < \beta$ let $G_\alpha : S_\omega(1+\alpha) \to Y$ be g-semihomogeneous. Then define $F : S_\omega(1+\beta) \to \kappa$ by

$$F(A) = \begin{cases} \lambda_0 & \text{if } A = 0, \\ G_\alpha(A - \{\alpha\}) & \text{if } A \neq 0 \text{ and } \alpha = \max(A). \end{cases}$$

It is easily checked that F is f-semihomogeneous. □

Corollary 5.4. *Suppose A is an admissible set, $\alpha < h(A)$, and that there is a weakly compact cardinal $\lambda >$ card(A). Then there is $\beta < h(A)$ such that whenever $X \subseteq \kappa$ is a strongly β-stationary subset and* (i), (ii) *or* (iii) *of Theorem 4.3 is satisfied, then there is no κ-like α-ordered model of σ_1 which embeds X.*

Proof. This follows immediately from Theorem 4.3 and Proposition 5.3. For, let ϕ be a sentence of L_A which has an α-ordered model but no non-well-ordered model. In addition, we can assume that ϕ has arbitrarily large models and has no symbols in common with σ_1 except for $\preceq$. Now suppose that for each $\beta < h(A)$ there is κ and a strongly β-stationary subset $X \subseteq \kappa$ satisfying conditions (i), (ii) or (iii), such that σ_1 has a κ-like α-ordered model $\mathfrak{A}_\beta$ which embeds X. We may assume also that $\mathfrak{A}_\beta$ is a model of ϕ. By Theorem 4.3, $\sigma_1 \wedge \phi$ has a λ-like model which embeds a stationary subset $Y \subseteq \lambda$. But this contradicts Proposition 5.3. □

The previous corollary has a curious proof. First of all, it relies on the existence of a large cardinal; and secondly, the proof takes a circuitous route through the model-theoretic results of section 4. A more direct proof not relying on the existence of a large cardinal should be available. What's more, a proof of a stronger result, such as the converse to Theorem 5.8 would be desirable. For the finite portion of the stationary hierarchy this is done in the next theorem.

Theorem 5.5. *If $n < \omega$ and $X \subseteq \kappa$ is n-stationary, then σ_1 has no κ-like n-ordered model which embeds X.*

Before proceeding with the proof of the theorem, we need a lemma which generalizes Fodor's Theorem. It asserts that for any κ, the collection of subsets of κ which are not n-stationary forms a κ-complete normal ideal. Our stumbling block in extending Theorem 5.5 to the transfinite portion of the stationary hierarchy results from our inability to extend this lemma.

Lemma 5.6. *Suppose that $X \subseteq \kappa$ is n-stationary, where $n < \omega$, and that $f : X \to \kappa$ is regressive. Then there is $\nu < \kappa$ such that $\{\mu \in X : f(\mu) = \nu\}$ is an n-stationary subset of κ.*

Proof. We give a proof by induction on n, the case $n = 0$ being Fodor's Theorem. Suppose the lemma is true for $n = m$; we show it to be true when $n = m+1$.

Let $X \subseteq \kappa$ be $(m+1)$-stationary and let $f : X \to \kappa$ be regressive. Suppose for each $\nu < \kappa$, that $P_\nu = \{\mu \in X : f(\mu) = \nu\}$ is not $(m+1)$-stationary. Hence, $Y_\nu = \{\lambda < \kappa : P_\nu \cap \lambda$ is m-stationary$\}$ is not stationary. Thus there is closed unbounded $C_\nu \subseteq \kappa$ such that $C_\nu \cap Y_\nu = 0$. Let $C = \{\mu < \kappa : \mu \in C_\nu$ for all $\nu < \mu\}$, the so-called diagonal intersection of $\langle C_\nu : \nu < \kappa \rangle$. Thus, $C \subseteq \kappa$ is closed unbounded, so that there is $\lambda \in C$ such that $X \cap \lambda$ is m-stationary. By the induction hypothesis, there is $\nu < \lambda$ such that $P_\nu \cap \lambda$ is m-stationary. However, if $\nu < \lambda$, then $\lambda \in C_\nu$, and this contradicts the fact that $P_\nu \cap \lambda$ is m-stationary. □

Proof of Theorem 5.5. Again we prove the theorem by induction on n. The case $n = 0$ follows from Fodor's Theorem. Assume that the theorem is true when $n = m$; we will prove it for $n = m+1$.

Let $X \subseteq \kappa$ be $(m+1)$-stationary, $C \subseteq \kappa$ a closed unbounded set and $f : S_\omega(X \cap C) \to \kappa$ regressive. By the lemma, there is an $(m+1)$-stationary $Y \subseteq X \cap C$ such that $f(\nu) = f(\mu)$ for each $\mu, \nu \in Y$. Thus there is $\lambda < \kappa$ such that $Y \cap \lambda$ is m-stationary. Select $\lambda_0 \in Y$ such that $\lambda_0 > \lambda$. Define $g : S_\omega(Y \cap \lambda) \to \lambda$ by $g(A) = p(f(A), f(A \cup \{\lambda_0\}))$. Then g is regressive, so by the inductive hypothesis there is g-semihomogeneous $G : S_\omega(m+1) \to Y \cap \lambda$. Define $F : S_\omega(m+2) \to Y$ by

$$F(A) = \begin{cases} G(A - \{m+1\}) & \text{if } A \neq 0, \\ \lambda_0 & \text{if } A = 0. \end{cases}$$

F is easily seen to be f-semihomogeneous. □

A result similar to Corollary 5.4 can be given for the subtle hierarchy using the existence of, for example, a measurable cardinal. However, in the next theorem we give a purely combinatorial proof of a stronger result.

Theorem 5.7. *If $X \subseteq \kappa$ is $(\alpha+1)$-subtle, then σ_2 has no κ-like $(1+\alpha)$-ordered model which embeds X.*

Proof. We give a proof by induction on α. For $\alpha = 0$ the theorem is clear. Suppose that $\beta > 0$ is such that for each $\alpha < \beta$ the theorem is true. Let $X \subseteq \kappa$ be $(\beta+1)$-subtle, let $C \subseteq \kappa$ be a closed unbounded set, and let $f : S_\omega(X \cap C) \to \kappa$ be regressive. Since κ is inaccessible we can assume that C contains only infinite cardinals.

Now for each $\nu \in X \cap C$, let

$$S_\nu = \{p(A, f(A \cup \{\nu\})) : A \in S_\omega(X \cap C \cap \nu)\} ,$$

where p is the pairing function. There exists $\lambda \in X \cap C$ such that $Y = \{\nu \in X \cap C \cap \lambda : S_\nu = \nu \cap S_\lambda\}$ is β-subtle.

Now we can define a regressive function $g : S_\omega(Y) \to \lambda$ by $g(A) = f(A \cup \{\lambda\})$. Notice that, by the way Y is defined, whenever $\nu \in Y$ and $A \in S_\omega(Y \cap \nu)$, then $f(A \cup \{\nu\}) = g(A)$. If $\beta = \gamma + 1$ for some γ, then by the induction hypothesis there is a g-homogeneous $G : S_\omega(2+\gamma) \to Y$. Define $F : S_\omega(2+\beta) \to X \cap C$ by

$$F(A) = \begin{cases} G(A - \{2+\gamma\}) & \text{if } A \neq 0 , \\ \lambda & \text{if } A = 0 . \end{cases}$$

Clearly F is f-homogeneous.

If β is a limit ordinal, then whenever $\gamma + 2 < \beta$ there is a g-homogeneous function $G_\gamma : S_\omega(2+\gamma+1) \to Y$. Now define $F : S_\omega(2+\beta) \to X \cap C$ by

$$F(A) = \begin{cases} G_\gamma(A) & \text{if } A \neq 0 \text{ and } \max(A) = 2+\gamma, \\ \lambda & \text{if } A = 0 . \end{cases}$$

Again, F is f-homogeneous. □

Theorem 5.8. *If $X \subseteq \kappa$ is not α-stationary, then σ_1 has a κ-like α-ordered model which embeds X.*

Proof. We give a proof by induction on α. For $\alpha = 0$ the theorem is trivial. As an inductive hypothesis, suppose that $\gamma > 0$ and that for each $\alpha < \gamma$ the theorem is true. We will show that is true when $\alpha = \gamma$.

Let $X \subseteq \kappa$ not be γ-stationary. If γ is a limit ordinal, then there is some $\alpha < \gamma$ such that X is not α-stationary, so by the inductive hypothesis we are done. Thus, we can assume that $\gamma = \beta + 1$. Therefore, $H = \{\lambda < \kappa : X \cap \lambda$ is a β-stationary subset of $\lambda\}$ is not stationary. Let $C \subseteq \kappa$ be closed unbounded such that $C \cap H = 0$, and further suppose that C consists only of infinite cardinals.

We will prove that for each $\nu \in X \cap C$ there is a regressive function $f_\nu : S_\omega(X \cap C \cap \nu) \to \nu$ for which there is no f_ν-semihomogeneous function $G : S_\omega(1+\beta) \to X \cap C \cap \nu$. If $\langle f_\nu : \nu \in X \cap C\rangle$ were such a sequence, then we could define $f : S_\omega(X \cap C) \to \kappa$ by $f(A) = f_\nu(A - \{\nu\})$, where $\nu = \max(A)$. Then f is the desired function. For, if $F : S_\omega(1+\gamma) \to X \cap C$ is f-semihomogeneous, then define $G : S_\omega(1+\beta) \to X \cap C$ by $G(A) = F(A \cup \{1+\beta\})$. But then G is f_ν-semihomogeneous, where $\nu = F(0)$. This is a contradiction.

We now construct the sequence $\langle f_\nu : \nu \in X \cap C\rangle$ by induction on ν. The only real problem occurs when ν is a limit point of $X \cap C$ and is inaccessible. Since $\nu \notin H$, then $X \cap \nu$ is not a β-stationary subset of ν. Thus, by the inductive hypothesis, there is a closed unbounded $D \subseteq \nu$ and a regressive $g : S_\omega(X \cap D \cap \nu) \to \nu$ for which there is no semihomogeneous $G : S_\omega(1+\beta) \to X \cap D \cap C$. We can assume $D \subseteq C$.

We are now going to define a function f'_ν with domain $S_\omega(X \cap C \cap \nu)$. Let $A = \{\nu_0, \ldots, \nu_{n-1}\} \in S_\omega(X \cap C \cap \nu)$, where $\nu_0 < \ldots < \nu_{n-1}$. Let $\delta_i = \sup((D \cup \{0\}) \cap (\nu_i + 1))$, for $i < n$. Then $f'_\nu(A)$ is defined according to cases:

(i) $f'_\nu(A) = \langle 1, g(A)\rangle$ if $\delta_0 = \nu_0, \ldots, \delta_{n-1} = \nu_{n-1}$.

(ii) $f'_\nu(A) = \langle 2, \delta_0\rangle$ if $\delta_0 = \nu_0$ and $n = 1$.

(iii) $f'_\nu(A) = \langle 3, f_\mu(A)\rangle$, where $\mu = \min(C - (\nu_{n-1}+1))$, if $n > 1$ and $\delta_0 = \delta_1 = \ldots = \delta_{n-1}$.

(iv) $f'_\nu(A) = \langle 0, 0\rangle$ if none of (i), (ii) or (iii) is applicable.

Recall that p is the pairing function, and then define $f_\nu : S_\omega(X \cap C \cap \nu) \to \nu$ by $f_\nu(A) = p(f'_\nu(A))$. It is easily checked that f_ν is regressive and there is no f_ν-semihomogeneous function $G : S_\omega(1+\beta) \to X \cap C \cap \nu$. □

The following corollary should be compared with Theorem 4.1.

Corollary 5.9. *For each $n < \omega$ there is a sentence σ with the following property: If $X \subseteq \kappa$ then σ has a κ-like model which embeds X iff X is not n-stationary.*

Proof. Use Theorems 5.5 and 5.8. □

Theorem 5.10. *If $X \subseteq \kappa$ is not α-subtle, then σ_2 has a κ-like α-ordered model which embeds X.*

Proof. We give a proof by induction on α which parallels the proof of the previous theorem, although there do exist some technical complications. Again the case when $\alpha = 0$ and α is a limit ordinal can be immediately disposed of.

For the case $\alpha = 1$, suppose $X \subseteq \kappa$ is not 1-subtle, as demonstrated by the sequence $\langle S_\nu : \nu < \kappa \rangle$. For each $\lambda \in X$, let $Y_\lambda = \{\nu \in X \cap \lambda : S_\nu = \nu \cap S_\lambda\}$, and let $H = \{\lambda \in X : Y_\lambda$ is not 0-subtle$\}$. Thus there is a closed unbounded $C \subseteq \kappa$ such that $C \cap H = 0$, and for each $\lambda \in X \cap C$ there is a closed unbounded $D_\lambda \subseteq \lambda$ such that $Y_\lambda \cap D_\lambda = 0$. Notice that if $\nu, \lambda \in X \cap C$ are such that $\nu < \lambda$ and $S_\nu = \nu \cap S_\lambda$, then $D_\nu \neq \nu \cap D_\lambda$. For each $\lambda \in X \cap C$, let $Z_\lambda = \{\nu \cdot 2 : \nu \in S_\lambda\} \cup \{\nu \cdot 2 + 1 : \nu \in D_\lambda\}$. Then define regressive $f : S_\omega(X \cap C) \to \kappa$ so that whenever $\nu, \lambda \in X \cap C$ and $\nu < \lambda$, then $f(\{\nu, \lambda\}) = \delta(Z_\nu, \nu \cap Z_\lambda)$. The function δ is defined by

$$\delta(A, B) = \begin{cases} 1 + \min((A - B) \cup (B - A)), & \text{if } A \neq B, \\ 0 \quad , & \text{if } A = B. \end{cases}$$

It is easy to check that there is no f-homogeneous $F : S_\omega(2) \to X \cap C$.

To continue with the induction, suppose that X is not γ-subtle, where $\gamma = \beta + 1 > 0$, and suppose that the theorem has been proved when $\alpha = \beta$. Thus there is a sequence $\langle S_\nu : \nu \in X \rangle$ demonstrating that X is not γ-subtle. For $\lambda \in X$, let $Y_\lambda = \{\nu \in X \cap \lambda : S_\nu = \nu \cap S_\lambda\}$, and let $H = \{\lambda \in X : Y_\lambda$ is not β-subtle$\}$. Let $C \subseteq \kappa$ be closed unbounded such that $C \cap H = 0$, and further suppose that C consists only of limit cardinals. For each $\lambda \in X \cap C$, let $D_\lambda \subseteq \lambda$ be a closed unbounded set, and let $g_\lambda : S_\omega(Y_\lambda \cap D_\lambda) \to \lambda$ be a regressive function for which there is no g_λ-homogeneous function $G : S_\omega(1 + \beta) \to Y_\lambda \cap D_\lambda$. Let $Z_\lambda = \{\nu \cdot 2 : \nu \in S_\lambda\} \cup \{\nu \cdot 2 + 1 : \nu \in D_\lambda\}$.

We will prove that for each $\nu \in X \cap C$ there is a regressive function $f_\nu : S_\omega(X \cap C \cap \nu) \to \nu$ for which there is no f_ν-homogeneous function $G : S_\omega(1 + \beta) \to X \cap C \cap \nu$. The same construction as in the proof of Theorem 5.8 will provide us with the regressive $f : S_\omega(X \cap C) \to \kappa$ for which there is no f-homogeneous $F: S_\omega(1+\gamma) \to X \cap C$.

We construct the sequence $\langle f_\nu : \nu \in X \cap C \rangle$ by induction on ν, the only problem occurring when ν is a limit point of $X \cap C$ and is inaccessible We are first going to define a function f'_ν with domain $S_\omega(X \cap C \cap \nu)$. Let $A = \{\nu_0, \ldots, \nu_{n-1}\} \in S_\omega(X \cap C \cap \nu)$, where $\nu_0 < \ldots < \nu_{n-1}$. Let $\delta_i = \sup((D_\nu \cup \{0\}) \cap (\nu_i + 1)$, for $i < n$. Then $f'_\nu(A)$ is defined according to cases:

(i) $f'_\nu(A) = \langle 1, g_\nu(A)\rangle$ if $n > 2$ and $\delta_0 = \nu_0, \ldots, \delta_{n-1} = \nu_{n-1}$.
(ii) $f'_\nu(A) = \langle 2, \delta_0\rangle$ if $n = 1$ and $\delta_0 < \nu_0$.
(iii) $f'_\nu(A) = \langle 3, \delta(Z_{\nu_0}, Z_{\nu_1})\rangle$ if $n = 2$ and $\delta_0 = \nu_0$ and $\delta_1 = \nu_1$.
(iv) $f'_\nu(A) = \langle 4, f_\mu(A)\rangle$, where $\mu = \min(C - (\nu_{n-1} + 1))$, if $n > 1$ and $\delta_0 = \delta_1 = \ldots = \delta_{n-1}$.
(v) $f'_\nu(A) = \langle 0, 0\rangle$ if none of (i)–(iv) apply.

Recall that p is the pairing function. Then define $f_\nu : S_\omega(X \cap C \cap \nu) \to \nu$ by $f_\nu(A) = p(f'_\nu(A))$. It can readily be checked that f_ν is regressive and that there is no f_ν-homogeneous function $G: S_\omega(1+\beta) \to X \cap C \cap \nu$. □

There is a small gap between Theorems 5.7 and 5.10. It would be pleasant if there were a sentence σ with the property that σ has an α-ordered κ-like model embedding X iff $X \subseteq \kappa$ is not α-subtle. For the finite portion of the subtle hierarchy this is already accomplished by the sentence σ_2, thus demonstrating that our definition and Baumgartner's definition of n-subtle are equivalent.

Corollary 5.11. *For each $n < \omega$ there is a sentence σ with the following property: If $X \subseteq \kappa$ then σ has a κ-like model which embeds X iff X is not n-subtle.*

Proof. Use Theorems 5.7 and 5.10. □

6. Erdös cardinals

Using Erdös cardinals we can give some further examples of subtle subsets. For each ordinal α, define the Erdös cardinal $\kappa(\alpha)$ to be the least cardinal κ such that $\kappa \to (\alpha)_2^{<\omega}$ (that is, whenever $f: S_\omega(\kappa) \to 2$, then there is an f-homogeneous set of length α). We know from [22] that if α is a limit ordinal and $\lambda < \kappa(\alpha)$, then $\kappa(\alpha) \to (\alpha)_\lambda^{<\omega}$.

Theorem 6.1. *If α is a limit ordinal, $C \subseteq \kappa(\alpha)$ is closed unbounded, and $f : S_\omega(C) \to \kappa(\alpha)$ is regressive, then there is an f-homogeneous $X \subseteq C$ of length α.*

Proof. For each $\nu < \kappa(\alpha)$, let $h_\nu : S_\omega(\nu) \to 2$ have no homogeneous set of length α. For each $n < \omega$ let $f_n : C^{n+1} \to \kappa(\alpha)$ and $g_n : C^{n+1} \to \kappa(\alpha)$ be defined by: if $\nu_0 < \ldots < \nu_n < \kappa(\alpha)$, then $f_n(\nu_0, \ldots, \nu_n) = f(\{\nu_0, \ldots, \nu_n\})$ and $g_n(\nu_0, \ldots, \nu_n) = h_{\nu_n}(\{\nu_0, \ldots, \nu_{n-1}\})$. Let $\mathfrak{A}$ be a Skolemization of the structure $(\kappa(\alpha), <, C, f_0, f_1, \ldots, g_0, g_1, \ldots)$. Choose $b_0 \in C$ to be the least element $b \in C$ for which there is a set $X \subseteq C$ of indiscernibles for $\mathfrak{A}$ of length α such that $b = \min(X)$. Let $\{b_\nu : \nu < \alpha\}$ be a set of indiscernibles.

First, notice that each b_ν is a limit point of C; otherwise, $\{\max(b_\nu \cap C) : \nu < \alpha\}$ would also be indiscernible and $\max(b_0 \cap C) < b_0$. Now if $\{b_\nu : \nu < \alpha\}$ fails to be f-homogeneous, then there exists $m < \omega$ such that $f_m(b_0, \ldots, b_m) \neq f_m(b_{m+1}, \ldots, b_{2m+1})$. For each $\nu < \alpha$, let

$$d_\nu = f_m(b_{(m+1)\nu}, \ldots, b_{(m+1)\nu+m})$$

and let

$$c_\nu = \min(C - (d_\nu + 1)) .$$

Clearly, it must be that $d_\nu < d_\mu$ whenever $\nu < \mu < \alpha$, as otherwise there would be an infinite decreasing sequence of ordinals. If $c_0 < c_1$, then $\{c_\nu : \nu < \alpha\}$ is an indiscernible set of length α and $c_0 < b_0$, which is a contradiction. If $c_0 = c_1$, then $\{d_\nu : \nu < \alpha\}$ is h_{c_0}-homogeneous, which is also a contradiction. □

It easily follows from Theorem 6.1 that if α is a limit ordinal, then $\kappa(\alpha)$ is α-subtle. What is surprising is that we can do much better.

Theorem 6.2. *Suppose κ is such that for any closed unbounded $C \subseteq \kappa$ and any regressive $f : S_\omega(C) \to \kappa$ there is an infinite f-homogeneous set. Then κ is κ-subtle.*

Proof. Let $\gamma < \kappa$. We will show that κ is γ-subtle. Assume, to the contrary, that κ is not γ-subtle.

We will define by induction on $n < \omega$ sets $D_n \subseteq [\kappa]^n$. Simultaneously, we will define for each $A \in D_n$, the following: C^A, X^A, $\langle S^A_\nu : \nu \in X^A \rangle$ and β^A.

For $n = 0$ let $D_0 = \{0\}$. Now since κ is not γ-subtle, we can find $\beta < \gamma$, a closed unbounded $C \subseteq \kappa$ and $\langle S_\nu : \nu < \kappa \rangle$ such that whenever $\lambda \in C$, then $\{\nu < \lambda : S_\nu = \nu \cap S_\lambda\}$ is not β-subtle. Let $C^0 = C$, $X^0 = \kappa$, $S_\nu^0 = S_\nu$ and $\beta^0 = \beta$.

Now suppose we have defined $D_n \subseteq [\kappa]^n$. Also, suppose that for each $B \in D_n$ we have already defined C^B, X^B, S^B and β^B. Letting $\mu = \min(B)$, we will have that $C^B \subseteq \mu$ is closed unbounded, $X^B \subseteq \mu$ and $\beta^B < \gamma$. Furthermore, whenever $\lambda \in C^B \cap X^B$, then $\{\nu \in X^B \cap \lambda : S_\nu^B = \nu \cap S_\lambda^B\}$ is not a β^B-subtle subset of λ.

By induction, let

$$D_{n+1} = \{\{\lambda\} \cup B : B \in D_n \quad \text{and} \quad \lambda \in C^B \cap X^B\}.$$

Suppose $A = \{\lambda\} \cup B \in D_{n+1}$, where $\lambda \in C^B \cap X^B$. Then set

$$X^A = \{\nu \in X^B \cap \lambda : S_\nu^B = \nu \cap S_\lambda^B\}.$$

Since X^A is not β^B-subtle, we can find $\beta^A < \beta^B$, a closed unbounded $C^A \subseteq \lambda$ and $\langle S_\nu^A : \nu \in X^A \rangle$ such that whenever $\mu \in C^A \cap X^A$, then $\{\nu \in X^A \cap \mu : S_\nu^A = \nu \cap S_\mu^A\}$ is not a β^A-subtle subset of μ. (However, in case $\beta^B = 0$, then let $\beta^A = 0$, $C^A \subseteq \lambda$ be closed unbounded such that $C^A \cap X^A = 0$, and $S_\nu^A = 0$ for each $\nu \in X^A$.) For the case that $A \in S_\omega(\kappa) - D$, let $\beta^A = C^A = X^A = 0$.

Let $f, g, h : S_\omega(\kappa) \to \kappa$ be such that whenever $\nu_0 < \ldots < \nu_n < \kappa$, then

$$f(\nu_0, \ldots, \nu_n) = \beta^{\{\nu_0, \ldots, \nu_n\}},$$

$$g(\nu_0, \ldots, \nu_n) = \delta(C^{\{\nu_0, \ldots, \nu_{n-1}\}}, \nu_0 \cap C^{\{\nu_1, \ldots, \nu_n\}}),$$

$$h(\nu_0, \ldots, \nu_n) = \delta(S_{\nu_0}^{\{\nu_2, \ldots, \nu_n\}}, \nu_0 \cap S_{\nu_1}^{\{\nu_2, \ldots, \nu_n\}}).$$

(The function δ is defined in the proof of Theorem 5.10.) By Theorem 6.1 there exists an infinite set $I = \{a_0, a_1, \ldots\} \subseteq C$, where $a_0 < a_1 < \ldots$, which is homogeneous for f, g, and h.

We will now show by induction on n that $[I]^n \subseteq D$, and that whenever $A \in [I]^n$ and $a < \min(A)$, $a \in I$, then $a \in X^A$. For $n = 0$ this is trivial, so suppose it's true for some n. From the definition of g it easily follows that

$$C^{\{a_0, \ldots, a_{n-1}\}} = a_0 \cap C^{\{a_1, \ldots, a_n\}}.$$

But $C^{\{a_0, \ldots, a_{n-1}\}}$ and $C^{\{a_1, \ldots, a_n\}}$ are closed unbounded subsets of a_0 and

a_1 respectively (because $[I]^n \subseteq D$), so that $a_0 \in C^{\{a_1, \ldots, a_n\}}$. Thus, $\{a_0, \ldots, a_n\} \in D$, or, more generally, $[I]^{n+1} \subseteq D$.

From the definition of h it easily follows that

$$S_{a_0}^{\{a_2, \ldots, a_{n+1}\}} = a_0 \cap S_{a_1}^{\{a_2, \ldots, a_{n+1}\}},$$

so that $a_0 \in X^{\{a_1, \ldots, a_{n+1}\}}$. More generally, if $A \in [I]^{n+1}$, $a < \min(A)$ and $a \in I$, then $a \in X^A$.

Not only have we shown that $S_\omega(I) \subseteq D$, but that $C^A \cap X^A \neq 0$ for any $A \in S_\omega(I)$. But then $\beta_0 > \beta^{\{a_0\}} > \beta^{\{a_0, a_1\}} > \ldots$, which is a contradiction. □

The above proof can be modified to show that $\kappa(\omega)$ is not the first cardinal λ which is λ-subtle. In fact, it can be shown that $\{\lambda < \kappa(\omega) : \lambda$ is λ-subtle$\}$ is subtle. The usual types of improvements of this also hold.

7. Open problems

There are some problems left unsolved, most of which are rather technical. A short list of some of them is given here.

(1) What can be done to Theorem 4.3 in the direction of eliminating the hypotheses (i)–(iii)?

(2) Find a combinatorial proof of Corollary 5.4 which eliminates the existence of a weakly compact cardinal as an hypothesis.

(3) Find a sentence σ such that for each α, κ and $X \subseteq \kappa$, the sentence σ has a κ-like α-ordered model which embeds X iff X is not α-stationary. Will σ_1 do? (Cf. Corollary 5.4 and Theorem 5.8.)

(4) Find a sentence σ such that for each α, κ and $X \subseteq \kappa$, the sentence σ has a κ-like α-ordered model which embeds X iff X is not α-subtle. Will σ_2 work? (Almost, by Theorems 5.7 and 5.10.)

(5) Let κ be the least cardinal such that whenever $C \subseteq \kappa$ is closed unbounded and $f : S_\omega(C) \to \kappa$ is regressive, then there is an infinite f-semihomogeneous subset of C. How large is κ? For example, does $\kappa = \kappa(\omega)$, or is κ less than the first weakly compact cardinal?

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